FREQUENCY OF GEOMETRIC ILLUSTRATIONS IN MATHEMATICS TEXTBOOKS, GRADES 1-6

Marvin J. Murr

Educational Research and Development Section
Instructional Materials Reference Center
American Printing House for the Blind

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Introduction

This study is a frequency survey of elementary level (grades I-VI) textbook series* for illustrations of geometric significance. It is intended primarily as a reference source for developers of educational materials. It was accomplished through a page by page textual analysis and frequency compilation of the occurrences of various geometric concepts. These concepts include all those entities commonly accepted as the core of geometry--elements, plane and space figures, abstract concepts, theorems, and constructions. The research methodology utilized in their development follows.

Category Criteria

Implicit in any frequency count is the "something" to be counted. In this study, geometric concepts constitute the categories of accountable items. Point, line, circle, and similar triangles are such concepts and each represents a unique category.

A problem arises in category determination when a single illustration contains more than one concept. For example, should two adjacent similar triangles be counted as: 1) one

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*Modern School Mathematics, Houghton Mifflin Co., Boston, 1967--Books I-VI

Elementary School Mathematics, Addison-Wesley Publishing Company, Inc., Palo Alto, 1963--Books I-III, 1964--Books IV-VI.

occurrence of "similar triangles", 2) two occurrences of "triangle," 3) numerous occurrences of "vertex, line segment, angle, closed curve, and plane figure," 4) all or part of the above? There are two logical approaches. One is an examination of context; the other, a consideration of the authors' educational philosophy.

No illustration appears in a textbook without some kind of textual reference to it. The text will make reference to the illustration either directly or by implication. Often, this reference takes the form of a definition, or at least, a verbal identification. When a theorem is presented, all the accompanying statements and questions lead the student toward the grasp of some theoretical proposition. In every case, context identifies the main concept being taught. If two similar triangles are referred to as such, contextually, then it may be assumed that "similar triangles" is the exemplified concept.

The educational philosophy of the two textbook series is one of intuitive learning. Basically, this is a method of discovering arithmetic and geometric realities through guided participation, rather than dogmatic presentation.*

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*This method often results in the presentation of concepts, theorems, etc., without their formal nomination. Since it is assumed that a teacher would lead her students to some geometric conclusion (concept identification) with regard to these illustrations; these open-ended presentations have been classified for counting according to traditional geometric teaching.

This system builds complex concepts inductively on simpler ones. Instruction proceeds from points, line segments, and angles to triangles and congruent triangles. With this method, the student's thought processes become more and more flexible as a result of his "intuitive habit," and he may see parts of the whole as readily as the whole itself. From this viewpoint, every triangular illustration, for example, would also be an illustration of vertex, line segment, angle, etc., to the observant student.

Although both the argument from context and philosophy appear valid, one seems more sound, psychologically. An exceptional student may be aware of all constituent parts of, say, a triangle when it is presented in context as a "triangle," but the vast majority would probably take the simpler psychological step and perceive it as a whole, i.e., just a "triangle." The average student would probably receive some subliminal learning with regard to the "parts," but his main attention and concept reinforcement would be determined by the "whole" that is textually referenced.

A further development of this problem occurs when an illustration definitely reflects two or more concepts. This is hardly a problem in the primary grades (I-III) where illustrations and concepts almost always reflect a 1:1 ratio, because educational goals are restricted to simple identification. In the intermediate grades (IV-VI), however, these simple concepts are used in the development of more complex theorems. The evolution of a theorem may require the employment of parallel lines, intersecting lines, and angles--all in one illustration. Such an illustration is classified in accordance with the "wholistic approach", and recorded as a particular theorem. But, if no theorem is apparently inferred, but rather, two or more concepts for themselves, then each important concept is credited with occurrence. Decisions, in this regard, are based on contextual emphases, and represent the principal concept(s) presented.

Frequency Criteria

It has been established "what" was counted. "How" leads us to a second methodological problem: the criteria of "one occurrence." There may seem to be no problem here since we have already defined our categories, and all that is left to do is to count their frequencies. For the most part this is true; however, there remain some problems.

The problem of frequency criteria was first encountered in the counting of plane figures. Circle, triangle, square, rectangle, and diamond appear numerously in illustrations of arithmetic operation, namely addition, subtraction, multiplication, and division. Here they are utilized as counting objects. Although not the prime purpose of the illustrations as geometric concepts, their influence as such could hardly be described as subliminal. It is possible to count "unknown" objects; however, in the classroom, it is likely that children are counting "circles", "triangles", and "squares".

On the other hand, it would not be safe to assume that the concept of "circle" is reinforced with each circle counted in a set of 100 circles, nor when 50 circles are added to 50 more. In the latter case, the exact reinforcing power of each circle is indeterminate, but it is a reasonable supposition that the concept of "circle" was perceived at least twice, since the task was adding two groups of circles. For this reason, the five plane figures mentioned above were credited as "one occurrence" for every group in which they appeared, no matter what their exact frequency was in that group. This method resulted in as many tallies per group as there were different concepts represented.

These groups were most often delineated by enclosing curves or spatial separation on the page. If a figure appeared in a group, but in more than one color, size, or shape; it received an additional tally for each novel variation. Most figures did not vary in any of these dimensions, but if so, resulted in only one extra tabulation.

The frequency of other plane figures and concepts, as well as nominating cases of the above figures, was usually easily determined. Customarily, they were accompanied by antecedent numbers or letters, identifying names, or grouped as examples after appellative sentences or bold type headings.

The frequency criterion applied here was that each example, individually identified or different from others in any respect (grouped examples), constituted one occurence.

There were instances, as previously noted, of presentation of two or more concepts in a single illustration. In addition, one or more of these concepts was sometimes exemplified several times within such an illustration. For example, "angle" may have been demonstrated through use of a rectangle. Since "angle" was the main intent of the illustration, there would be reason for tallying four such cases of it. This was done if each was individually identified in the figure or text; otherwise, only one occurrence was noted.

Presentations of constructions were significantly different and deserved special treatment. Though three or more illustrations were involved in the development of one construction, a single occurrence was recorded. Actually, these were three parts of one illustration, and additional tallies would create a false impression.

Data Sources

Data has been drawn from sources found throughout each textbook. However, certain textbook sections supplied a greater variety, if not also a greater number, of geometric illustrations than others. These sections were the geometry chapters and lessons found within the analyzed texts.

Each book of the Houghton Mifflin series (H-M) contained a chapter dealing with geometry. The Addison-Wesley series (A-W) had similar chapters in its fifth and sixth grade texts, but had only 2-page geometry lessons following most arithmetic chapters in third and fourth grade books. There was no explicit geometry instruction in its first or second grade textbooks.

In a few cases, geometric chapters and lessons contained proportionately fewer illustrations than were found to occur in other, regular, arithmetic sections. These cases occurred in the primary level textbooks where circles, triangles, squares, and diamonds were extensively used as counting objects or in the illustration of basic arithmetic operations. Here, geometric illustrations occurred more frequently than in the chapters and lessons on geometry, but with much less variety.

Glossaries of terms, found only in H-M's books III through VI, also contained a number of geometric illustrations. But since it is doubtful that they would comprise a formal part of an instructional program, they have been omitted from this analysis.

Data Divisions

Collected data in this study represent 236 separate geometric concepts. This amount of information needs classification if it is to be useful. But *how* should this be done? There are research advantages to presentations based upon chronological appearance, alphabetical order, and numerical frequency. However, we are mainly interested in which concepts appear at each grade level and their relative frequencies. Types of appearing concepts is also a matter of importance. Therefore, all concepts encountered have been divided under eight descriptive headings which summarily signify typical characteristics of each concept category. Under these headings, concepts are listed, first, by chronological appearance (grade level), and secondly, in order of decreasing frequency. This system contains the desirable advantages of chronological appearance and numerical frequency as well as the grouping of similar types of concepts together for easy comparison and reference.

A brief outline of these categories is introduced here, along with their nominating criteria. A complete set of criteria will be given as each category is presented in the following pages.

Geometric Concepts

I. Elements Basic, tangible geometric concepts.

II. Plane Figures Two-dimensional simple closed curves.

III. Space Figures Three-dimensional closed surfaces.

IV. Abstract Concepts Concepts of geometric relationship.

V. Constructions Geometric drawings and model building.

VI. Theorems and Postulates Proven geometric propositions and accepted assumptions.

I. Elements

Elements are most easily understood by observing their relationship to other geometric concepts. They are the "raw" material" out of which plane and space figures, and constructions are formed. Though not as elementary as sets of points, they are the basic geometric concepts on which all others are founded. Elements are the concrete entities by which abstract concepts and theorems are exemplified, though not by a simple additive process. They differ from abstract concepts in that they have real existence in themselves. A "line segment" (element) can be seen and touched, but the reality of "parallel lines" (abstract concept) lies in their relationship. Table 1 presents these elements by grade level and textbook series.

Elements occur frequently (1825) throughout the texts and are surpassed only by the prodigious appearance of plane figures (5835). The most numerous figures are "points" (644), "line segments" (469), and "angles" (181). "Points" constitute more than 1/3 of all element occurrences and "line segments" more than 1/4. If "points on number lines" (771, See Table 10) were included here, "points" alone would outnumber all the other elements combined.

Noteworthy observations of Table 1 include:

1) "Line segment" is the most frequently appearing element in each of the primary grades, while "point" has similar status in the intermediate grades.

TABLE 1

Geometric Elements

Grades	1		2		3		4		5		6		
Series	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	Total
Point*		24	8	7	57	9	198	4	111	70	29	127	644
Line Segment	41		61	21	49	43	81	49	29	53	16	26	469
Not Cl Curve	7		10		9								26
Side (poly)			19		3		3		11		15		51
Curve			16		15		10		2		1		44
Line			3		11		7		2	7	3	1	34
Simple Curve			18										18
End Point			1		7								8
Not Simple Curv			7										7
Angle					13	1	15		38	68	18	28	181
Ray					23		15		3	12	3	4	60
Right Angle					25	1	6		7	1	2		42
Radius					8	6	9		6		2		31
Vertex (poly)					2		4	1	11		5		23
Diagonal					6			1	4		7		18
Diameter					6	1	4		4		1		16
Plane					1		9		2	2	1	1	16
Center pt (cir)					4	7	1		2				14
Vertex (angle)					1		5		1	1	1		9
Surface					2						5		7
Chord						1			1		1		3
Base							4		4		15		23
Face							2	3	4		6	2	17
Edge							2	2	4		5	3	16
Vertex (sp fig)							3	2	4		4	2	15
Not a Plane							4						4
Sides (angle)							2		1		1		4
Legs (rt tri)								2				1	3
Central Angle								2					2
Inscribed Angle								2					2
Hypotenuse								1				1	2
Arc									1		1	1	3
Bisector (angl)											3		3
Bisector (l.s.)											3		3
Curved Surface											2		2
Perp Bist (l.s.)											2		2
Straight Angle											1	1	2
Sides (match)											1		1
Subtotals	48	24	143	28	242	69	384	69	252	214	154	198	(1223 602)
Totals	72		171		311		453		466		352		1825

*Also see "points on number lines," Table 10.

2) The overall frequency of element occurrence rises with increasing grade level through grade V, then falls off in grade VI. Whether this downward trend continues in junior high and high school texts is unknown, but seems likely since elements are rudimentary geometric concepts.

3) It is generally true that once an element appears, it is also found in every subsequent grade. (This is the case for most of the concepts in this study.)

4) Elements occur about twice as often in the H-M series as in the A-W series (1223 to 602).

II. Plane Figures

Plane figures were defined above as "two-dimensional simple closed curves." A simple closed curve is one which has no ends and does not cross itself. This category would include all simple closed curve figures which lie in one plane; but, in this study, most often refers to one of the basic geometric figures--square, circle, triangle, etc.

Plane figures constitute the single largest category of concepts in this analysis (5835). Their great number is not surprising since they are presented not only in and for themselves, but also--and mainly--as items used to demonstrate arithmetic operations and counting.

Table 2 is a frequency distribution of plane figures. Not included here is the entry, "plane figure," since it represents the concept, and no particular object. The five most frequently occurring concepts have been totalled (indicated by parentheses) by grade level to allow easy comparison between concepts over grade levels.

Table 2 shows that "circle" is the most frequently occurring concept (2516), followed in order by "triangle" (1261), "square" (964), "rectangle" (385), and "diamond" (299). These five figures are so plentiful (93% of total) mainly because they are the chief figures used in exercises demonstrating arithmetic operations and counting. This use would account for the greater frequencies in the primary grades where these operations are taught.

The occurrence of plane figures is greatest in the first grade and diminishes as we advance to the sixth grade level (with a minor exception in grade V). "Circle" is the single most frequently appearing figure at all six grade levels. It occurs twice as often as the second-ranking "triangle", and constitutes nearly half of the total plane figure frequency. Each textbook series was found to contain an approximately equal number of plane figures.

III. Space Figures

Space figures are three-dimensional geometric figures usually occurring as closed surfaces. They range from the familiar "sphere" and "pyramid" to the little known "torus", which resembles an auto inner tube. Their frequency is sparce (536) compared to their two-dimensional cousins--plane figures. Table 3 reports the frequencies of space figures.

TABLE 2

Plane Figures

Grades	1		2		3		4		5		6		Total
Series	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	
	(497)		(451)		(668)		(331)		(393)		(176)		
Circle	196	301	242	209	294	374	219	112	346	47	146	30	2516
	(373)		(239)		(133)		(209)		(207)		(100)		
Triangle	224	149	156	83	70	63	96	113	169	38	77	23	1261
	(309)		(184)		(162)		(111)		(129)		(69)		
Square	157	152	32	152	64	98	41	70	92	37	52	17	964
	(91)		(144)		(59)		(32)		(38)		(21)		
Rectangle	15	76	11	133	12	47	9	23	9	29	10	11	385
	(88)		(37)		(54)		(37)		(53)		(30)		
Diamond	54	34	26	11	22	32	20	17	33	20	27	3	299
Pentagon		33		15	2	6	2	8	4	1	1	1	73
Hexagon	6	23		21	1	6		2	2		2	1	64
Closed Curve	18		1		17		5		1		4		46
Ellipse	1				5		5		3	5	1		20
Parallelogram					9		6	4	13	6	7		45
Quadrilateral					4	3	4	2	4	2	9	1	29
Sim Cl Curve					2		8	1	2	2	4		19
Rt Triangle						1		1	8		2	3	15
Polygon							6	14	14	12	15	9	70
Trapezoid								1		3		1	5
Octagon								1	1	1	1		4
Nt Sim Cl Curv							3						3
Circum Circle								1					1
Inscrb Circle								1					1
Isos Rt Tri								1					1
Isos Triangle									1		3	4	8
Equil Tri									1		1		2
Septagon									1				1
Great Circle											2		2
Rhombus											1		1
Subtotals	671	768	468	624	502	630	424	372	704	203	365	104	(3134 2701)
Totals	1439		1092		1132		796		907		469		5835

TABLE 3

Space Figures

Grades	1		2		3		4		5		6		
Series	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	Total
Cube		6			17	51	10	76	13	19	22	10	224
Sphere		18		5	6		7		3	3	1	2	45
Cylinder					8		3		5	3	1	2	22
Simp Cl Surface							26		23		22	5	76
Cone							2		2	3		2	9
Tetrahedron								2	3		2		7
Pyramid							2				4		6
Sq Pyramid								1	2		2		5
Plane Angle							2						2
Rect Prism									27	13	29	18	87
Triang Prism									4	2	3	2	11
Rect Pyramid									1	3	1	2	7
Square Prism										4			4
Hemisphere										2			2
Hex Pyramid										2			2
Torus										2			2
Truncated Cone										2			2
Pentag Pyramid									1		1		2
Prism											12		12
Reg Polyhedron											5		5
Dodecahedron											1		1
Hexahedron											1		1
Icosahedron											1		1
Octahedron											1		1
Subtotals		24		5	31	51	52	79	84	58	109	43	(276 260)
Totals	24		5		82		131		142		152		536

It is immediately apparent from Table 3 that space figures are mainly intended for presentation at the intermediate level of geometric instruction. There are a total of 111 space figures presented in the primary grades, while 425 appear at the intermediate levels. The more frequent appearance of space figures at the intermediate level seems to reflect their advanced nature and complexity. This fact is not surprising since spheres, cubes, and pyramids could not be properly appreciated without a prior knowledge of circle, square, and triangle.

The space figure of highest frequency is the "cube". It occurs 224 times and at every grade level, except second. "Rectangular prism" ranks second (87) and "simple closed surface" third (76). "Sphere" is the only space figure represented at each grade level and places fourth in frequency.

Although ranking fourth, "sphere" is the most frequently occurring space figure in the first two grades (the only grade II entry). "Cube" outranks all other space figures in grades III and IV, while "rectangular prism" is most frequent in grades V and VI. In a comparison of textbook frequency, space figures appear in almost identical numbers (H-M's 276 to A-W's 260).

IV. Abstract Concepts

Concepts included in this category are those of geometric relationship. They are illustrated by means of elements and plane and space figures, but do not have concrete existence in themselves. "Symmetry" is a good example. This entity cannot be physically encountered, or isolated from the components between or in which it exists. It is simply a relational concept. Previously encountered concepts, such as angle (an element), circle (a plane figure), and sphere (a space figure) can be conceived solely in abstract terms, but each of these concepts also has a concrete, material form which physically represents the abstract idea. In the case of abstract concepts defined here, no such corresponding physical realities exist.

Other concepts in this group are part abstract and part physical concepts. "Parallel lines" and "congruent angles" are representative of these compound ideas. "Lines" and "angles" can be physically represented in geometry, while "parallel" and "congruent" are purely conceptual ideas. Thus, "parallel lines" and "congruent angles" have been placed in this category, rather than previous ones--elements, particularly--because their abstract portion appeared from context to constitute their most important aspect. It is the "parallel" and "congruent" features of these compounds which appeared foremost. Abstract concepts are presented in Table 4.

An inspection of Table 4 reveals that "region" (529) is by far the most frequently occurring abstract concept. It far exceeds the second ranking "space figure" (67) and third place concept, "points, on, inside, or outside a simple closed curve" (49). Adding to its pre-eminence is the fact that "region" is the abstract concept of greatest frequency at every grade level.

TABLE 4

Abstract Concepts

Grades	1		2		3		4		5		6		
Series	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	Total
Region	28		57		67		86	93	67	80	26	25	529
POIOS* Cl Crv	17		20		6		3		1			2	49
Triang Region			3		3		10		5			13	34
Rect Region			5		4		9		5		6		29
Square Region			8		4		5		3		4		24
Circular Region			17		4		1						22
Cong Curves			7										7
Intersection			2										2
Shortest Distance			1								1		2
Symmetry					19		9		4		1		33
Cong Angles					5				6	3	5	8	27
Parallel Lines					7		2	1	1		1	2	14
Asymmetry					8		3						11
Parallel L.S.'s					3		4		2				9
Ins & Outs Cir						3							3
In, Outs. Qadrl						3							3
Endpts On a LS					2								2
Pts On A Curve					2								2
Pts On A Line					2								2
Nonparall Lns					2								2
Ins & Outs Tri						1		1					2
Ins, Outs Angls						1							1
Ins, Outs Rt Ang						1							1
Cong L.S.'s							13		4	5	2	7	31
Ln of Symmetry							6		13		7		26
Cross Section							1		1		12	9	23
Intrsectng L.S.'s							12						12
Intrsectng Lns							4				4	3	11
Ln of Nonsymmetry							9						9
Parallel Planes							5		2		2		9
Geo Set of Pts								8					8
POIOS* Cl Surf							6				1		7
Perp Lines							3				1	1	5
Height							2						2
Nonparallel L.S.'s							2						2
Space Figure									11	48		8	67
Similar Triangls									18		10	4	32
Plane Figure									9	2	8	7	26
Cong Triangles									3	6	4	11	24
Noncong Triangls										4		7	11

TABLE 4 (continued)

Grades	1		2		3		4		5		6		
Series	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	Total
Cong Sides									5		1		6
Noncong L.S.'s										4		1	5
Parallel Sides									3				3
Perp L. S.'s									3				3
Intersect. Planes									2		1		3
Inter of Ln & Pl									1				1
Perp Rays									1				1
Polygonal Region											20		20
Linear Pair											4		4
Rigid											4		4
Noncong Angle												2	2
Per Ln's & L.S's											2		2
Distance											1		1
Similarity											1		1
Skew Lines											1		1
Subtotals	45		120		138	9	195	103	170	152	130	110	(798-374)
Totals	45		120		147		298		322		240		1172

*Signifies--Points On, Inside, or Outside a Simple...

As we proceed up through the grades, the occurrence of abstract concepts increases, reaching a peak in the fifth grade. This trend is understandable since these concepts are increasingly abstract and require the geometric sophistication of the intermediate student.

Abstract concepts appear approximately twice as often in the H-M textbook series (798) as they do in the A-W series (374). Since these abstract concepts reflect conceptual depth in geometry, the H-M elementary series would appear more advanced than its counterpart, the A-W series.

V. Constructions

Constructions, in geometry, signify the drawing of geometric elements and plane figures. However, the building of space figure models has also been included here since it involves a construction process.

Construction of elements and plane figures is accomplished through the mediation of paper, pencil, ruler, compass, and protractor. Most space models are made by copying a "plan" from the textbook onto paper or cardboard, cutting it out, and fastening it together. As stated above, a single complete construction may involve three or more illustrations; however, since only one constructed figure results, one occurrence was noted.

Constructions have been divided into the three concept areas which they encompass: elements, plane figures, and space figures. In Tables 5, 6, and 7, these constructions are listed

TABLE 5

Element Constructions

Concepts	Source	Total
Parallel line segments	3-1, 5-3	4
Center pt of a "given" circle	3-1	1
Diameter	III-1	1
Chord	III-1	1
Right Angle	3-2, III-4, 4-2, VI-2	10
Parallel lines	IV-5, VI-1	6
Diagonals in polygons	IV-3	3
Centrl & inscrib angls in a cir	IV-1	1
Perpendicular bisector	IV-1, 6-1, VI-1	3
Center point	IV-1	1
Angle	5-1, 6-1	2
Bisecting an angle	V-1, 6-1, VI-1	3
Bisecting a line segment	V-1	1
Copying a "given" ln segment	V-1	1
Copying a "given" angle	V-1, VI-1	2
Testing for congr in ln segs	V-1	1
Testing for congr in angles	V-4, VI-4	8
A perpendicular to a pt on a ln	6-1, VI-2	3

Grades	1	2	3	4	5	6	Total
H-M			4	2	4	4	14
A-W			6	11	9	12	38
Total			10	13	13	16	52

TABLE 6

Plane Figure Constructions

Concepts	Source	Total
Circle	3-2, III-3, 4-1, IV-1, 5-2	9
Square	III-1	1
Right triangle	III-2	2
Circle through a "given" point	III-4	4
Circle through two "given" points	III-3	3
Circle through 3 "given" points	III-3	3
Circle with a "given" radius	III-1	1
Triangle	III-1	1
Quadrilateral	III-1	1
Circle with a "given" diameter	III-1	1
Ellipse	4-1, 5-2, 6-2,	5
Geometric figures on a number plane	4-2, 5-4	6
Parallelogram	IV-3, 5-1	4
Inscribed circle in a triangle	IV-1	1
Cir circmscrb about a "given" tri	IV-1	1
Isosceles right triangle	IV-1	1
Parabola	IV-1	1
Regular hexagon	5-1, 6-1	2
Rectangle	5-1	1
Triangle with "given" sides	5-1, 6-1, VI-1	3
Triangle "given" 2 angls & side bet	5-1, VI-1	2
Copying a "given" circle	V-1	1
Triangle congruent to a "given" one	V-2, 6-2, VI-3	7
Rectangle with 4 "given" sides	6-1	1
Equilateral tri with side "given"	6-1	1
Triangle given 2 sides & angle bet	VI-1	1

Grades	1	2	3	4	5	6	Total
H-M			2	4	13	8	27
A-W			20	8	3	6	37
Total			22	12	16	14	64

TABLE 7

Space Figure Constructions

Concepts	Source	Total
Model pyramid (square)	4-1, VI-1	2
Model cube	4-1, IV-1, 5-1, V-1, VI-1	5
Model cone	4-1, 5-1	2
Model sphere	4-2, 5-1	3
Model triangular pyramid (tetrahedron)	IV-2, 5-1	3
Model rectangular prism	5-1, V-1, VI-1	3
Model triangular prism	5-1	1
Model cylinder	5-1, V-1	2
Model hexagonal prism	VI-1	1
Model octahedron	VI-1	1

Grades	1	2	3	4	5	6	Total
H-M				5	7		12
A-W				3	3	5	11
Total				8	10	5	23

in order of first appearance in the H-M series, by grade level. When a construction appeared in the A-W series, it was recorded wherever it was found in the H-M list. If it was not contained there, then it was chronologically inserted after the last H-M entry for that grade level. This method was adopted to conserve information which had been accidentally collected, and may be of use to some researcher.

From the tables, it may be observed that no constructions appear at the first or second grade levels. However, they do appear in all following grades, and with approximately equal frequency (32, 33, 39, 35).

The most frequently appearing element, and plane and space figure constructions are the "right angle" (10), "circle" (9), and "cube" (5), respectively. While the constructions of elements and plane figures are found at the third grade level, space figures are not represented till the fourth grade. Space figures (23) also occur considerably less frequently than do constructions of elements (52) and plane figures (64). On the whole, constructions occur next to last in concept frequency (Theorems and Postulates) and more often in the A-W series (A-W's 86 to H-M's 53).

VI. Theorems and Postulates

Postulates are basic, *a priori* assumptions underlying the entire realm of geometry. They are the "accepted" identities and definitions which constitute the starting point for geometric study. Without them, there would be no bases for further geometric development into elements, figures, concepts, and theorems.

Theorems are proven geometric propositions. They are somewhat like "discovered" relationships which are found to be consistently true. "The sum of distances from the foci of an ellipse to the curve, itself, is constant" is an example. This relationship has been proven many times and is now accepted as fact.

Since postulates and theorems are so closely related, they are presented together without discrimination. The method of presentation observed in the section on "Constructions" has also been utilized in Table 8. Again, concepts are listed by grade level as they appear in the H-M series. Novel concepts, encountered only in the A-W series, are listed subsequently. Numbers in parentheses represent grade totals.

Table 8 shows that theorems and postulates occur only in grades three through six. Their absence in the first two grades can be interpreted as a reflection of their more advanced nature. This is further borne out by the increasing frequencies encountered in higher grades, with a culmination in grade six almost twice any previous grade level frequency.

Table 8 also shows that theorems and postulates are represented by 39 concept categories containing 133 total frequencies. Of these 39 categories, 34 occur less than five times each.

TABLE 8

Theorems and Postulates

	Concepts	Source
1.	The intersection of a line with parallel lines forms sets of congruent angles [and linear pairs (a)].	3-1, 5-2, 6-3a, VI-6
2.	The perpendicular distance between parallel lines is constant.	3-2, 4-4, VI-2
3.	The three angles of a triangle equal a straight angle.	3-1, III-1, VI-2
4.	The four angles of a quadrilateral equal two straight angles.	3-1, III-2
5.	A diagonal of a parallelogram [rectangle (a)] divides it into two congruent triangles.	3-1, IV-1, 5-1a, VI-1a
6.	Intersecting lines form congruent angles [and linear pairs (a)].	3-1, 6-1a, VI-1
7.	The two acute angles of a right triangle equal a right angle.	III-1
8.	When the distance between the centers of congruent intersecting circles is equal to their radii, lines joining them to each other and to the points of intersection are congruent.	4-1
9.	Three points (not in one line) define a plane.	4-2, 6-2
10.	Intersecting diagonals of a square form right angles.	4-1
11.	Two lines perpendicular to the same line are parallel.	4-1, 5-1, VI-2
12.	A line through the centers of congruent circles and a tangent line common to all are parallel.	4-1
13.	Concentric circles have the same center point.	4-1
14.	The size and shape of an ellipse is determined by the distance between its foci, and the value of the constant combined distance between each foci and the ellipse.	4-1
15.	The intersection of a line and plane is a point.	4-1, 5-1
16.	The intersection of two planes is a line [line segment (a)].	4-7, 4-3a, 5-1, 6-1

TABLE 8 (continued)

17.	The intersection of two planes forms an angle.	4-1
18.	A line and a point (not on that line) define a plane.	4-1, 5-1
19.	Two lines may define a plane.	4-1
20.	Three planes may intersect at a point.	4-3, 6-1
21.	Three planes may intersect to form a line.	4-2, 6-1
22.	Parallel planes intersect a cylinder to form congruent intersections.	4-1
23.	Pythagorean Theorem	4-1, IV-1, 5-4, 6-22, VI-4
24.	Two inscribed angles equal a central angle when each subtends a similar arc.	IV-1
25.	Triangles inscribed in a circle with its diameter as their common base form right angles at the circumference.	5-1
26.	Lines of symmetry cut geometric figures into two congruent parts.	5-2
27.	The intersection of two sets of parallel lines forms congruent sets of angles and line segments [and linear pairs (a)].	5-3, 6-2, 6-2a
28.	The sum of distances from the foci of an ellipse to the curve, itself, is constant.	5-2
29.	Any polygon may be divided into a set of triangles by drawing diagonals.	5-1
30.	When lines are drawn between the centers of two intersecting circles, and between the centers and points of intersection, two isosceles triangles are formed.	5-1
31.	Triangles formed within a circle by two intersecting diameters and two chords are congruent.	5-1
32.	The length of angle rays is not related to angle size.	V-1
33.	Only one line can be drawn parallel to another, through a given point.	6-1
34.	The shortest distance between a line and a point is a perpendicular line from the line to the point.	6-1
35.	A quadrilateral inscribed in a circle with its diagonal the diameter of the circle and dividing the quadrilateral into two congruent parts - is a rectangle.	6-1

36.	Two intersecting lines define a plane.	6-2
37.	All points in a plane equidistant from a given point, constitute a circle.	VI-1
38.	A line, perpendicular to one of two parallel lines, is perpendicular also to the other.	VI-1
39.	Points on concentric circles nearest to and farthest from a given point (not the center) constitute a line.	VI-3

1		2		3		4		5		6		Sub-Total		
H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	Total
				(11)		(36)		(23)		(63)				
				7	4	33	3	22	1	40	23	102	31	133

Among the five remaining categories, the "Pythagorean Theorem" occurs most often with 32 frequency cases.

Symbolic Plane Figures

The plane figures in this group are identical to four concepts listed in the "Plane Figures" section--circles, triangles, squares, and rectangles. However, the plane figures included here were used symbolically to represent arithmetic operations, answer spaces, and content indicators.

A triangle within a problem indicated that an arithmetic operation was called for, such as addition or subtraction, and that the student place the appropriate symbol (+ or -) into this triangular answer space. The square and circle were answer spaces for numbers and equality signs, respectively.

Rectangles were used where squares would not be large enough to accommodate an answer. Additionally, a circle, a square, or a triangle appeared at the top of each page in the H-M series as indicators of content: regular text, test, and extra work, respectively.

In all cases, the geometric figures *signified something beyond themselves*. It is likely that they were not encountered in and for themselves, but rather, as symbols of particular functions. It would appear, then, that they did not serve as figural learning reinforcements to the same extent as other textual occurrences of these figures. Table 9 presents these figures and their frequencies.

TABLE 9

Symbolic Plane Figures

Grade	1		2		3		4		5		6		
Series	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	Total
Place Holder Sq	188*	124*	177*	87*					1*				577*
Equality Circle	9*	14*	14*	13*	12*		14*		8*		9*		93*
Place Hold Rect	5*	10*	13*	17*									45*
Operational Tri	5*		9*		1*		2*		3*		1*		21*
Page Circle					306		304		300		304		1214
Page Square					11		11		11		10		43
Page Triangle					11		11		11		10		43

*Pages where the concept occurred more than once.

It is difficult to assess this data since frequencies are listed both in pages and individual counts. However, it does reinforce the popularity found to exist among the plane figures: circle, square, triangle, and rectangle.

Number Lines

A number line is a line on which specified points are labeled for the demonstration of arithmetic concepts and operations. As a whole, the number line is not a geometric concept. It does, however, contain or is made up of geometric concepts which "appear" identical to some which have been previously encountered: lines, line segments, points, and points on lines. If they were, indeed, identical to these already established concepts, a classification based on established criteria could be made for them. Though a number line appears to exemplify the concepts of line, point, etc. as they have been used above, these designations are technically incongruent with the concepts found in number lines.

Although called "number lines" in the textbooks, they actually appeared there most often as number line segments. It may be "understood" that they continue indefinitely in either direction as lines do, but they were usually illustrated as starting and ending at points, just as line segments appear. Since it is likely that these misnomers would lead a student to a distorted and false understanding of these concepts, lines and line segments appearing in number lines were left uncounted altogether.

The same reasoning may be applied to "points on number lines." At first sight, they may appear to be identical to the concept, "points on" a line, recorded above. However, "points on number lines" have the connotation of relative, quantitative position, while "points on" a line represent concepts of positional identity. The similarity of these two concepts lies only in their physical representation. For all these reasons, number lines and points on them have been treated as separate categories. Of the two, only "points on number lines" was counted as a geometric concept.

Number lines usually appeared in a one to one ratio with problems or exercises; but even when two or more accompanied a single problem, no two were identical illustrations. Therefore, each number line encountered was recorded as one occurrence.

Points on number lines appeared with such frequency that it is doubtful that each occurrence was consciously noticed. These points were often accompanied by numbers but usually appeared as unit markers with no special significance. A great number of points were individually referred to as "a point on the number line" and specified by a letter. Only these latter points were recorded, and along with number lines, are presented in Table 10.

Table 10 shows that the total number line frequency in each textbook series is approximately equal. However, the individual distributions vary and are inversely related.

H-M's primary grades contain twice as many number lines as does A-W's, while A-W's intermediate grades excel H-M's in number by more than one hundred. The great frequency of number lines employed in H-M's primary texts indicates an emphasis on such tools for the teaching of basic arithmetic concepts.

TABLE 10

Number Lines

Grades		1	2	3	4	5	6	Total
Number Lines	H-M	104	182	103	100	74	103	666
	A-W	20	101	73	175	143	84	596
		124	283	176	275	217	187	1262
Points on (-) Number Lines*	H-M		31-3		38-9	7-6	10-2	86-20
	A-W				290-114	235-88	160-48	685-250
			31-3		328-123	242-94	170-50	771-270

*These number lines are included in "Number Line" frequencies.

Although the distribution of number lines is spread almost evenly over H-M's six grades, the A-W series tends to concentrate them in its intermediate books. Here, they appear twice as often as they do in primary textbooks. Apparently, they are seen to be of more educational value in these higher grades.

"Points on number lines" are not individually specified until grade IV (except in H-M's grade II). Hereafter, they appear in the H-M series, but are greatly emphasized in each A-W intermediate text. Altogether, the A-W series presents eight times as many points on 12 1/2 times as many number lines.

Omissions

In much the same category as secondary plane figures, are a host of extensively appearing concepts which probably have only subliminal effect. Examples include line segments which divide the page, closed curves which group sets of items for arithmetic operations, rectangles used to separate tables, etc., from the rest of the text, periods, etc. All these appear too frequently to count, and are of questionable value in the inculcation of the specified concepts.

Observations

Previous discussions have focused largely on intracategorical frequency comparisons, over grade levels and textbook series. Although category totals across all grades and series have been mentioned in the treatment of each category, it would be advantageous to formally view these data. In addition, total frequencies across all concept categories, by grade level and series, would be useful in gaining a full overview of all the study statistics.

In Table 11, main category frequencies have again been presented as handy references for the "total" entries and as vehicles for the simultaneous viewing of all the study data (except data in Tables 9 and 10).

TABLE 11

Data Summary

Grades	1		2		3		4		5		6		Sub-Totals		Total
Series	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	H-M	A-W	
Elements	(72)		(171)		(311)		(453)		(466)		(352)				
	48	24	143	28	242	69	384	69	252	214	154	198	1223	602	1825
Plane Figures	(1439)		(1092)		(1132)		(796)		(907)		(469)				
	671	768	468	624	502	630	424	372	704	203	365	104	3134	2701	5835
Space Figures	(24)		(5)		(82)		(131)		(142)		(152)				
		24		5	31	51	52	79	84	58	109	43	276	260	536
Abstract Concepts	(45)		(120)		(147)		(298)		(322)		(240)				
	45		120		138	9	195	103	170	152	130	110	798	374	1172
Theorems & Post.					(11)		(36)		(23)		(63)				
					7	4	33	3	22	1	40	23	102	31	133
Constructions					(32)		(33)		(39)		(35)				
					6	26	11	22	24	15	12	23	53	86	139
Subtotals	764	816	731	657	926	789	1099	648	1256	643	810	501	5586	4054	
Totals	1580		1388		1715		1747		1899		1311				9640

The data of greatest significance in Table 11 are the frequency totals. These parameters can be used to identify overall trends in the collected data. A simple comparison of category frequencies reveals that plane figures (5835) are, by far, the most numerous type of illustrations, though elements (1825) and abstract concepts (1172) still appear quite often. Space figures (536) occur about half as frequently as abstract concepts, while constructions (139) and theorems and postulates (133) are relatively rare on the average.

The frequency of geometric concepts increases until grade six, where a substantial decrease is noted. The causes for this steady rise lie in the occurrence of larger numbers of elements, abstract concepts, space figures, and theorems and postulates. Constructions remain about constant in the upper grades while plane figures decrease. Since plane figures are used most heavily in the illustration of arithmetic operations, it is not surprising that their frequencies drop off in the grades where this goal is no longer sought. Decreasing numbers of these figures is the prime cause of the frequency dip observed at the sixth grade level.

A comparison of frequencies in textbook series shows that the H-M series surpassed its competitor at every grade level, except first, and in every concept category, except constructions. Overall, the H-M books contain about 27% more geometric illustrations (5586 to 4054) than does the A-W series.

Summary

This study has been a textbook analysis for illustrations of geometric significance. Geometric concepts were identified and a frequency count of their appearance in illustrations was compiled. Plane figures were found to constitute over one-half of all illustrated concepts, with elements and abstract concepts contributing substantial portions to the total number.

A trend of increasing concept frequency was noted with progression through grade levels. Its causes were increasing frequency levels of elements, space figures, abstract concepts, and theorems and postulates. Plane figures decreased with increasing grade levels while constructions remained constant, once they appeared.

When the H-M textbook series was compared to the A-W series for concept frequency, the H-M series was found to contain more illustrations in every concept category, except constructions, and at every grade level, except first. Of particular significance were the categories of "elements" and "abstract concepts", where the H-M textbooks contained more than twice the number of the respective A-W frequencies, and "theorems and postulates," where this ratio reached treble proportions.